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# An exact and explicit treatment of an elliptic hole problem in thermopiezoelectric media

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**Abstract**

This paper presents an exact solution for the problem of an elliptic hole or a crack in a thermopiezoelectric solid. First, based on the extended version of Eshelby–Stroh's formulation, the generalized 2D problems of an elliptical hole in a thermopiezoelectric medium subject to uniform heat flow and mechanical–electrical loads at infinity are studied according to exact boundary conditions at the rim of the hole. The complex potentials in the medium and the electric field inside the hole are obtained in closed form, respectively. Then, when the hole degenerates into a crack, the explicit solutions for the field intensity factors near the crack tip and the electric field inside the crack are presented. It is shown that the singularities of all the field are dependent on the material constants, the applied heat load and mechanical loads at infinity, but not on the applied electric loads. It is also found that the electric field inside the crack is linearly variable, which is different from the result based on the impermeable crack model. 

*Keywords:* Piezoelectric solid; Hole; Crack; Thermal analysis; Exact solution

## 1. Introduction

During the past few decades, widespread attention has been given to the thermal stress problems in an elastic medium with inclusions, holes or cracks. A considerable work on this subject can be found in the literature. For example, one can cite the work of Florence and Goodier (1960), Sih (1962), Lee and Jang (1993), Zhang and Hasebe (1993), Chao and Shen (1993, 1997), Kattis and Patia (1994), and Kaminskii and Flegantov (1994) for the cases of isotropic media, and also those of Sturla and Barber (1988), Hwu (1990, 1992), Tarn and Wang (1993), Chao and Chang (1994), Lin et al. (1997) and Chao and Shen (1998) for the cases of anisotropic materials. In recent years, the thermo-electric-mechanical coupling problem in thermopiezoelectric media with holes or cracks has also received much attention with increasingly wide application of thermopiezoelectric materials in the engineering. In contrast, however, relatively little work has

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been done due to the mathematical complexity. Recently, several solutions of thermopiezoelectric media with cracks have been given by Shang et al. (1996) for the 3D problem of a penny-shaped crack, and also by Yu and Qin (1996), Qin and Mai (1997), Yang et al. (1997), Shen and Kuang (1998) and Qin et al. (1999) within the framework of 2D analysis. But it should be seen that all the above analyses are based on an impermeable boundary assumption, i.e. the electric field inside cracks is assumed to be zero. More and more findings show the assumption may lead to erroneous results for the crack problem in piezoelectric media (see, e.g. the work of McMeeking (1989), Pak and Tobin (1993), Dunn (1994), Sosa and Khutoryansky (1996), Kogan et al. (1996), Zhang et al., 1998 and Gao and Fan (1998, 1999)). More recently, Gao and Wang (2001) studied the 2D problem of thermopiezoelectric materials with cracks by means of the Parton assumption, i.e. the crack is considered as a thin slit and thus the normal components of electric displacement and the tangential component of electric field are assumed to be continuous across the slit (Parton, 1976). However, the correctness of Gao and Wang's results (2001) remain to be proved. Thus, it is very necessary to give an exact and explicit solution for a crack in thermopiezoelectric media. It is well known that an elliptic hole problem is the basis of the corresponding crack problem in elastic analysis. Although the exact solution of a crack in a linear piezoelectric solid has been obtained by Gao and Fan (1999) and Gao (2000) who began with an elliptical hole, to the authors' knowledge, the similar success has yet not been reached for the crack problem in thermopiezoelectric solid.

In the present work, we treat the generalized 2D problem of an elliptic hole or a crack in an infinite thermopiezoelectric medium subjected to uniform heat flow together with uniform mechanical-electric loads at infinity. The analysis is based on the Stroh formalism and the exact boundary conditions at the rim of the hole. The whole contents consist of five sections. Following this brief introduction, basic equations concerning the thermopiezoelectricity are summarized in Section 2. Then, both the analytical solutions and numerical results of the elliptic hole are presented in Section 3, respectively. In Section 4 given are the exact solutions of a crack, including the complex potential in the medium, the field intensity factor near the crack tips and the electric field inside the crack. Finally, the conclusions on the current work are drawn in Section 5.

## 2. Basic equations

In a fixed rectangular coordinate system $x_j$ $(j = 1, 2, 3)$, denoting by $\mathbf{u}$, $\varphi$, $\sigma$, $\mathbf{D}$, $\mathbf{E}$, $T$ and $q$ the displacements, electric potential, stress, electric displacement, electric field, temperature and heat flux, respectively, the complete set of governing equations for piezothermoelastic problem can be expressed, in the stationary case without body force, extrinsic bulk charge and heat source, as (Mindlin, 1974; Wu, 1984; Chandrasekharaiah, 1988; Shen and Kuang, 1998)

$$\sigma_{ij} = c_{ijkl}\gamma_{kl} - e_{ijs}E_s - \beta_{ij}T \tag{1}$$

$$D_i = \varepsilon_{is}E_s + e_{irs}\gamma_{rs} + \tau_i T \tag{2}$$

$$q_i = -\lambda_{ij}T_{,j} \tag{3}$$

$$\gamma_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{4}$$

$$E_i = -\varphi_{,i} \tag{5}$$

$$\sigma_{ij,j} = 0 \tag{6}$$

$$D_{i,i} = 0 \tag{7}$$

$$q_{i,i} = 0 \tag{8}$$

where

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}, \quad e_{kij} = e_{kji}, \quad \varepsilon_{ij} = \varepsilon_{ji}, \quad \beta_{ij} = \beta_{ji}, \quad \lambda_{ij} = \lambda_{ji}$$

In the above equations, repeated indices imply summation, a comma stands for differentiation, and $c_{ijkl}$, $e_{kij}$, $\varepsilon_{ij}$, $\beta_{ij}$, $\lambda_{ij}$, $\tau_i$ are the elasticity constants, piezoelectricity constants, dielectric constants, stress-temperature coefficients, coefficients of heat conduction and pyroelectric coefficients, respectively.

Substituting (1), (2) and (3) together with (4) and (5) into (6), (7) and (8), respectively, one has

$$\left(c_{ijrs}u_r + e_{sji}\varphi\right)_{,si} - \beta_{ij}T_{,i} = 0 \tag{9}$$

$$\left(-\varepsilon_{is}\varphi + e_{irs}u_r\right)_{,si} + \tau_i T_{,i} = 0 \tag{10}$$

$$\lambda_{ij}T_{,ij} = 0 \tag{11}$$

Consider the generalized 2D problems of thermopiezoelectricity with geometry and loading independent of $x_3$. In this case, the governing Eq. (11) becomes

$$\lambda_{11}\frac{\partial^2 T}{\partial^2 x_1^2} + 2\lambda_{12}\frac{\partial^2 T}{\partial^2 x_1 x_2} + \lambda_{22}\frac{\partial^2 T}{\partial^2 x_2^2} = 0 \tag{12}$$

The general solution of (12) is

$$T = 2\mathrm{Re}\left[g'(z_t)\right], \quad z_t = x_1 + \mu_t x_2 \tag{13}$$

where Re means taking the real part; $g$ is a complex function to be determined; the prime (′) indicates differentiation with respect to its argument, and $\mu_t$ is the heat eigenvalue which is determined from

$$\lambda_{22}\mu_t^2 + 2\lambda_{12}\mu_t + \lambda_{11} = 0 \tag{14}$$

(14) produces the solution of $\mu_t$ with positive imaginary part as

$$\mu_t = (-\lambda_{12} + \mathrm{i}\kappa_t)/\lambda_{22} \tag{15}$$

$$\kappa_t = \left(\lambda_{11}\lambda_{22} - \lambda_{12}^2\right)^{1/2}, \quad \lambda_{11}\lambda_{22} - \lambda_{12}^2 > 0 \tag{16}$$

where $\mathrm{i} = \sqrt{-1}$.

Inserting (13) into (3), and then using (14)–(16) leads to

$$q_1 = 2\mathrm{Re}\left[\mathrm{i}\mu_t\kappa_t g''(z_t)\right] \tag{17}$$

$$q_2 = -2\mathrm{Re}\left[\mathrm{i}\kappa_t g''(z_t)\right] \tag{18}$$

On the other hand, the resultant heat flow $Q$ can be expressed as

$$Q = \int q_n \,\mathrm{d}s \tag{19}$$

where $s$ is the arc-length, $q_n$ stands for the heat flux in the direction normal to $s$.

Noting

$$q_n \,\mathrm{d}s = q_1 \,\mathrm{d}x_2 - q_2 \,\mathrm{d}x_1 \tag{20}$$

one has, by substituting (20) together with (17) and (18) into (19), that

$$Q = 2\mathrm{Re}\left[\mathrm{i}\kappa_t g'(z_t)\right] \tag{21}$$

From (19)–(21) one obtains

$$2\mathrm{Re}\left[\mathrm{i}g'(z_t)\right] = \frac{1}{\kappa_t}\int q_1\,\mathrm{d}x_2 - q_2\,\mathrm{d}x_1 \tag{22}$$

or

$$2\mathrm{Re}\left[\mathrm{i}g'(z_t)\right] = \frac{1}{\kappa_t}\int q_n(s)\,\mathrm{d}s \tag{23}$$

Introduce two function vectors:

$$\mathbf{u} = (u_1, u_2, u_3, \varphi)^t, \quad \phi = (\phi_1, \phi_2, \phi_3, \phi_4)^t$$

where the superscript $t$ represents the transpose; $\mathbf{u}$ and $\phi$ are generalized displacement function and generalized stress function, respectively, which are related to the field variables by

$$\sigma_{j1} = -\phi_{j,2}, \qquad \sigma_{j2} = \phi_{j,1} \quad (j = 1, 2, 3) \tag{24}$$

$$D_1 = -\phi_{4,2}, \quad D_2 = \phi_{4,1}, \quad E_1 = -u_{4,1}, \; E_2 = -u_{4,2} \tag{25}$$

Then, the general solution of $\mathbf{u}$ and $\phi$ can be written as

$$\mathbf{u} = \mathbf{u}_h + \mathbf{u}_p \tag{26}$$

$$\phi = \phi_h + \phi_p \tag{27}$$

where $\mathbf{u}_p$ and $\phi_p$ are the particular solution of (9) and (10), while $\mathbf{u}_h$ and $\phi_h$ the homogeneous solutions of (9) and (10) corresponding to the isothermal case, here $\mathbf{u}_h$ and $\phi_h$ can be expressed as (Suo et al., 1992; Chung and Ting, 1996)

$$\mathbf{u}_h = \mathbf{A}\mathbf{f}(z_*) + \overline{\mathbf{A}\mathbf{f}(z_*)} \tag{28}$$

$$\phi_h = \mathbf{B}\mathbf{f}(z_*) + \overline{\mathbf{B}\mathbf{f}(z_*)} \tag{29}$$

with

$$\mathbf{f}(z_*) = [f_1(z_1), f_2(z_2), f_3(z_3), f_4(z_4)]^t, \qquad z_\alpha = x_1 + \mu_\alpha x_2 \quad (\alpha = 1 \sim 4)$$

In (28) and (29), $\mathbf{A}$ and $\mathbf{B}$ are two $4 \times 4$ matrices, $f_\alpha(z_\alpha)$ are complex potentials to be found, and $\mu_\alpha$ are the complex eigenvalues with positive imaginary parts and can be obtained from the equation

$$|\mathbf{D}_*(\mu)| = \mathbf{0} \tag{30}$$

where

$$\mathbf{D}_*(\mu) = \mathbf{S} + \mu(\mathbf{R} + \mathbf{R}^t) + \mu^2\mathbf{W}$$

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_0 & \mathbf{e}_{11} \\ \mathbf{e}_{11}^t & -\varepsilon_{11} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \mathbf{R}_0 & \mathbf{e}_{21} \\ \mathbf{e}_{12}^t & -\varepsilon_{12} \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} \mathbf{W}_0 & \mathbf{e}_{22} \\ \mathbf{e}_{22}^t & -\varepsilon_{22} \end{bmatrix}$$

and

$$(\mathbf{S}_0)_{ik} = c_{i1k1}, \; (\mathbf{R}_0)_{ik} = c_{i1k2}, \; (\mathbf{W}_0)_{ik} = c_{i2k2}, \; (i, k = 1, 2, 3)$$

$$\mathbf{e}_{ik} = (e_{i1k}, e_{i2k}, e_{i3k})^t, \; (i, k = 1, 2)$$

In this paper we assume that $\mu_\alpha$ are distinct. For this case, $\mathbf{A}$ and $\mathbf{B}$ are nonsingular, and there is the following orthogonality relation (Chung and Ting, 1996):

$$\begin{bmatrix} \mathbf{B}^t & \mathbf{A}^t \\ \overline{\mathbf{B}}^t & \overline{\mathbf{A}}^t \end{bmatrix} \begin{bmatrix} \mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{B} & \overline{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{31}$$

where $\mathbf{I}$ is the $4 \times 4$ unit matrix.

On the other hand, the particular solutions of (9) and (10) are (Shen and Kuang, 1998)

$$\mathbf{u}_p = 2\mathrm{Re}[\mathbf{c}g(z_t)] \tag{32}$$

$$\phi_p = 2\mathrm{Re}[\mathbf{d}g(z_t)] \tag{33}$$

where $\mathbf{c}$ and $\mathbf{d}$ are the heat eigenvectors, which can be determined from the following equations

$$\mathbf{D}_*(\mu_t)\mathbf{c} = \beta_1 + \mu_t\beta_2$$

$$\mathbf{d} = (\mathbf{R}^t + \mu_t\mathbf{W})\mathbf{c} - \beta_2$$

$$\beta_1 = (\beta_{11}, \beta_{21}, \beta_{31}, \tau_1)^t$$

$$\beta_2 = (\beta_{12}, \beta_{22}, \beta_{32}, \tau_2)^t$$

Substituting (28), (29), (32) and (33) into (26) and (27) gives the final solution of $\mathbf{u}$ and $\phi$ such that

$$\mathbf{u} = 2\mathrm{Re}[\mathbf{A}\mathbf{f}(z_*) + \mathbf{c}g(z_t)] \tag{34}$$

$$\phi = 2\mathrm{Re}[\mathbf{B}\mathbf{f}(z_*) + \mathbf{d}g(z_t)] \tag{35}$$

Assuming that the considered problem satisfies such a condition that for an arbitrary point on the boundary, the corresponding points $z_t$ and $z_\alpha$ ($\alpha = 1$–4) can be translated into an identical point, e.g. on the $x_1$-axis or an unit circle, and as a result the boundary equation can be reduced to that containing one variable. Only under this condition, the one-complex-variable approach introduced by Suo (1990) can be used to simplified analysis when one considers the boundary conditions (Ting, 2000; Gao, 2001). In the present work these one-complex variable equations can be summarized as

$$T = 2\mathrm{Re}[g'(z)] \tag{36}$$

$$q_1 = 2\mathrm{Re}[\mathrm{i}\mu_t\kappa_t g''(z)] \tag{37}$$

$$q_2 = -2\mathrm{Re}[\mathrm{i}\kappa_t g''(z)] \tag{38}$$

$$Q = 2\mathrm{Re}[\mathrm{i}g'(z)] = \frac{1}{\kappa_t}\int q_1\,\mathrm{d}x_2 - q_2\,\mathrm{d}x_1 \tag{39}$$

$$\mathbf{u} = 2\mathrm{Re}[\mathbf{A}\mathbf{f}(z) + \mathbf{c}g(z)] \tag{40}$$

$$\phi = 2\mathrm{Re}[\mathbf{B}\mathbf{f}(z) + \mathbf{d}g(z)] \tag{41}$$

If the traction, and the normal component of electric displacement $D_n$ are given on the boundary, the corresponding boundary condition can be expressed as

$$2\mathrm{Re}[\mathbf{B}\mathbf{f}(z) + \mathbf{d}g(z)] = \int_s \mathbf{t}\,\mathrm{d}s, \quad \mathbf{t} = [t_1, t_2, t_3, D_n]^t \tag{42}$$

where $t_j$ ($j = 1, 2, 3$) is the component of surface traction vector.

After the solutions of $g(z)$ and $\mathbf{f}(z)$ are obtained from Eqs. (36)–(42), a replacement of $z_t$, $z_1$, $z_2$, $z_3$ or $z_4$ should be made for each component function to calculate field quantities from (24) and (25).

## 3. The solution to an elliptic hole

Consider a generalized 2D problem of a thermopiezoelectric medium containing an elliptic hole L, which is described by the equation: $(x_1^2/a^2) + (x_2^2/b^2) = 1$, as shown in Fig. 1. The uniform mechanical–electric loads $\Pi_2^\infty = \left(\sigma_{21}^\infty, \sigma_{22}^\infty, \sigma_{23}^\infty, D_2^\infty\right)^t$ and $\Pi_1^\infty = \left(\sigma_{11}^\infty, \sigma_{12}^\infty, \sigma_{13}^\infty, D_1^\infty\right)^t$ together with uniform heat flow $\mathbf{q}^\infty = \left(q_1^\infty, q_2^\infty\right)^t$ are simultaneously applied at infinity. In addition, the hole is assumed to be free of force, external charge and heat flow, but filled with air or vacuum.

### 3.1. The electric field inside the hole

Let the electric potential $\varphi_h(z)$ inside the hole be

$$\varphi_h(z) = 2\mathrm{Re} f_h(z) \tag{43}$$

where $f_h(z)$ is an analytic function. Then inside $L$, the electric field components $\left(E_1^0, E_2^0\right)$ and electric displacement components $\left(D_1^0, D_2^0\right)$ can be expressed as:

$$E_1^0 = -2\mathrm{Re}F_h(z), \quad E_2^0 = 2\mathrm{Im}F_h(z) \tag{44}$$

$$D_1^0 = -2\varepsilon_0\mathrm{Re}F_h(z), \quad D_2^0 = 2\varepsilon_0\mathrm{Im}F_h(z) \tag{45}$$

where Im indicates the imaginary part; $\varepsilon_0$ is the dielectric constant of air or vacuum; $F_h(z) = \mathrm{d}f_h(z)/\mathrm{d}z$.

Using Gauss' law, one has

$$\int_s D_n^0 \mathrm{d}s = \int_s D_2^0 \mathrm{d}x_1 - D_1^0 \mathrm{d}x_2 \tag{46}$$

Inserting (45) into the right side of (46) gives

$$\int_s D_n^0 \mathrm{d}s = 2\varepsilon_0 \mathrm{Im} f_h(z) \tag{47}$$

Noting that the exterior of the ellipse $L$ can be mapped onto the exterior of the unit circle $\gamma$ in the $\zeta$-plane by

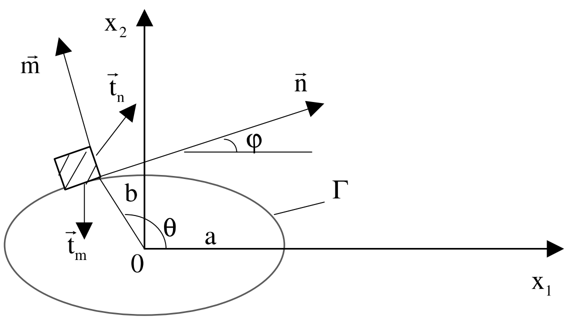


Fig. 1. An elliptical hole in a thermopiezoelectric solid.

$$z(\zeta) = R_0(\zeta + m_0\zeta^{-1}), \quad R_0 = (a+b)/2, \quad m_0 = (a-b)/(a+b)$$

then $f_h(z)$ can be expressed inside $L$ in the form of the Faber series as (Cutis, 1971; Kosmodamianskii and Chemie, 1981)

$$\hat{f}_h(\zeta) = \sum_{n=1}^{\infty} a_n^0(\zeta^n + m_0^n\zeta^{-n}) \tag{48}$$

where $\hat{f}_h(\zeta) = f_h[z(\zeta)]$; $a_n^0$ are complex coefficients to be determined.

### 3.2. The temperature field in the medium

From (39), the insulated boundary condition at the rim of the hole requires

$$2\mathrm{Re}[\mathrm{i}g'(z)] = 0 \tag{49}$$

For the present problem, $g'(z)$ takes the form of

$$g'(z) = c_t^{(2)}z + g_0'(z) \tag{50}$$

where $g_0'(z)$ is a holomorphic function outside the hole, and $g_0'(\infty) = c_t^{(1)}$ here $c_t^{(1)}$ is a constant corresponding to an uniform temperature field and thus can be neglected without loss in generality; $c_t^{(2)}$ is another constant to be determined.

Substituting (50) into (37) and (38), and then taking the limiting $z \to \infty$ yields

$$2\mathrm{Re}[\mathrm{i}\mu_t\kappa_t c_t^{(2)}] = q_1^{\infty} \tag{51}$$

$$2\mathrm{Re}[\mathrm{i}\kappa_t c_t^{(2)}] = -q_2^{\infty} \tag{52}$$

(51) and (52) give

$$c_t^{(2)} = \frac{q_1^{\infty} + \overline{\mu}_t q_2^{\infty}}{\mathrm{i}\kappa_t(\mu_t - \overline{\mu}_t)} \tag{53}$$

In fact, $c_t^{(2)}z$ in (50) stands for the complex potential of an infinite medium without hole subjected to the given uniform heat flow at infinity. Thus, one has from (39) that

$$2\mathrm{Re}[\mathrm{i}c_t^{(2)}z] = \frac{1}{\kappa_t}\int (q_1^{\infty}\,\mathrm{d}x_2 - q_2^{\infty}\,\mathrm{d}x_1) = q_s \tag{54}$$

where

$$q_s = \frac{1}{\kappa_t}(q_1^{\infty}x_2 - q_2^{\infty}x_1) \tag{55}$$

Substituting (50) into (49) and then using (54) results in

$$2\mathrm{Re}[\mathrm{i}g_0'(z)] = -q_s \tag{56}$$

Introduce the following mapping function $z_\alpha(\zeta)$:

$$z_\alpha(\zeta) = R_\alpha(\zeta + m_\alpha\zeta^{-1}) \tag{57}$$

with

$$R_\alpha = (a - \mathrm{i}\mu_\alpha b)/2, \quad m_\alpha = (a + \mathrm{i}\mu_\alpha b)/(a - \mathrm{i}\mu_\alpha b), \quad (\alpha = t; 1\text{–}4)$$

which transforms the exterior of the ellipse $L_\alpha$ in the $z_\alpha$-plane into the exterior of a unit circle $\gamma$ in $\zeta$-plane. Then, (56) can be rewritten as

$$2\mathrm{Re}\left[\mathrm{i}\hat{g}_0'(\zeta)\right] = -q_s \tag{58}$$

Noting that on the hole, $\zeta = \sigma = e^{\mathrm{i}\theta}$ and

$$x_1 = a\cos\theta = \frac{a}{2}\left(\frac{1}{\sigma} + \sigma\right) \tag{59}$$

$$x_2 = b\sin\theta = \mathrm{i}\frac{b}{2}\left(\frac{1}{\sigma} - \sigma\right) \tag{60}$$

one obtains by using (58) and (55), (59) and (60) that

$$2\mathrm{Re}\left[\mathrm{i}\hat{g}_0'(\sigma)\right] = -q_s(\sigma) \tag{61}$$

where

$$q_s(\sigma) = -\frac{1}{2\kappa_t}\left[aq_2^\infty\left(\sigma + \frac{1}{\sigma}\right) + \mathrm{i}bq_1^\infty\left(\sigma - \frac{1}{\sigma}\right)\right] \tag{62}$$

Multiplying both sides of (61) by $\int_\gamma \mathrm{d}\sigma/(\sigma - \zeta)$, and then calculating the Cauchy integration leads to (Muskhelishvili, 1975)

$$\mathrm{i}\hat{g}_0'(\zeta) = \frac{1}{2\kappa_t}\left[aq_2^\infty - \mathrm{i}bq_1^\infty\right]\zeta^{-1} + \mathrm{i}c_t^{(1)} \tag{63}$$

Using (63) and (50), the final form of $g'(z)$ can be expressed as

$$g'(z) = c_t^{(2)}z + c_t^{(1)} + \frac{1}{2\mathrm{i}\kappa_t}\left[aq_2^\infty - \mathrm{i}bq_1^\infty\right]\zeta^{-1}(z) \tag{64}$$

The integration of (64) with respect to $z$ gives

$$g(z) = \tfrac{1}{2}c_t^{(2)}z^2 + c_t^{(1)}z + \gamma_1\ln\zeta(z) + \gamma_2\zeta^{-2}(z) \tag{65}$$

where

$$\gamma_1 = \frac{R_t}{2\mathrm{i}\kappa_t}\left[aq_2^\infty - \mathrm{i}bq_1^\infty\right], \quad \gamma_2 = \frac{1}{2}m_t\gamma_1 \tag{66}$$

### 3.3. The electro-elastic field in the medium

Observing (65), the complex potential in the medium can be expressed as

$$\mathbf{f}(z) = \tfrac{1}{2}\mathbf{c}^{(2)}z^2 + \mathbf{c}^{(1)}z + \delta\ln\zeta(z) + \mathbf{f}_0(z) \tag{67}$$

where $\mathbf{f}_0(z)$ is a holomorphic function outside the hole; $\mathbf{c}^{(2)}$, $\mathbf{c}^{(1)}$ and $\delta$ are three constant vectors to be found.

To find $\delta$, one has to consider the force equilibrium condition and the conditions of single-valued displacement and electric potential. These conditions require

$$\oint_{\Gamma_n} \mathbf{u}_{,1}\,\mathrm{d}z = \mathbf{0}, \qquad \oint_{\Gamma_n} \phi_{,1}\,\mathrm{d}z = \mathbf{0} \tag{68}$$

where $\Gamma_n$ stands for a clockwise closed-contour encircling the hole, and

$$\mathbf{u}_{,1} = 2\mathrm{Re}\left[\mathbf{A}\mathbf{f}'(z) + \mathbf{c}g'(z)\right] \tag{69}$$

$$\phi_{,1} = 2\mathrm{Re}\left[\mathbf{B}\mathbf{f}'(z) + \mathbf{d}g'(z)\right] \tag{70}$$

Substituting (69) and (70) into (68), and then using the residue theorem produces

$$[\mathbf{A}\delta + \mathbf{c}\gamma_1] - \overline{[\mathbf{A}\delta + \mathbf{c}\gamma_1]} = \mathbf{0} \tag{71}$$

$$[\mathbf{B}\delta + \mathbf{d}\gamma_1] - \overline{[\mathbf{B}\delta + \mathbf{d}\gamma_1]} = \mathbf{0} \tag{72}$$

(71) and (72) show that $\mathbf{A}\delta + \mathbf{c}\gamma_1$ and $\mathbf{B}\delta + \mathbf{d}\gamma_1$ are real, respectively.

Using (31) one can obtain from (71) and (72) that

$$\delta = \mathbf{B}^t(\overline{\mathbf{c}\gamma_1} - \mathbf{c}\gamma_1) + \mathbf{A}^t(\overline{\mathbf{d}\gamma_1} - \mathbf{d}\gamma_1) \tag{73}$$

On the other hand, substituting (65) and (67) into (69) and (70), and then taking the limiting $z \to \infty$ (in this case, $\zeta \to \infty$) leads to

$$2\mathrm{Re}\left[\left(\mathbf{A}\mathbf{c}^{(2)} + \mathbf{c}c_t^{(2)}\right)z\right] + 2\mathrm{Re}\left[\mathbf{A}\mathbf{c}^{(1)} + \mathbf{c}c_t^{(1)}\right] = \mathbf{u}_{,1}^{\infty} \tag{74}$$

$$2\mathrm{Re}\left[\left(\mathbf{B}\mathbf{c}^{(2)} + \mathbf{d}c_t^{(2)}\right)z\right] + 2\mathrm{Re}\left[\mathbf{B}\mathbf{c}^{(1)} + \mathbf{d}c_t^{(1)}\right] = \phi_{,1}^{\infty} \tag{75}$$

where

$$\phi_{,1}^{\infty} = \Pi_2^{\infty}, \quad \mathbf{u}_{,1}^{\infty} = \left(\varepsilon_{11}^{\infty}, \varepsilon_{12}^{\infty} + \omega_3^{\infty}, 2\varepsilon_{13}^{\infty}, -E_1^{\infty}\right)^t \tag{76}$$

In (76), $\varepsilon_{11}^{\infty}$, $\varepsilon_{12}^{\infty}$, $\varepsilon_{13}^{\infty}$ and $E_1^{\infty}$ are the components of strain and electric field at infinity, respectively; $\omega_3^{\infty}$ denotes the rotation at infinity.

Considering the fact that both the stresses and strains are bounded at infinity, (74) and (75) gives

$$2\mathrm{Re}\left[\mathbf{A}\mathbf{c}^{(1)} + \mathbf{c}c_t^{(1)}\right] = \mathbf{u}_{,1}^{\infty} \tag{77}$$

$$2\mathrm{Re}\left[\mathbf{B}\mathbf{c}^{(1)} + \mathbf{d}c_t^{(1)}\right] = \phi_{,1}^{\infty} \tag{78}$$

and

$$2\mathrm{Re}\left[\left(\mathbf{A}\mathbf{c}^{(2)} + \mathbf{c}c_t^{(2)}\right)z\right] = \mathbf{0} \tag{79}$$

$$2\mathrm{Re}\left[\left(\mathbf{B}\mathbf{c}^{(2)} + \mathbf{d}c_t^{(2)}\right)z\right] = \mathbf{0} \tag{80}$$

Using (31) one obtains from (77) and (78) that

$$\mathbf{c}^{(1)} = \mathbf{B}^t\mathbf{u}_{,1}^{\infty} + \mathbf{A}^t\phi_{,1}^{\infty} - \mathbf{B}^t\left[\mathbf{c}c_t^{(2)} + \overline{\mathbf{c}c_t}^{(2)}\right] - \mathbf{A}^t\left[\mathbf{d}c_t^{(2)} + \overline{\mathbf{d}c_t}^{(2)}\right] \tag{81}$$

On the other hand, (79) and (80) imply that the complex functions $\left[\mathbf{A}\mathbf{c}^{(2)} + \mathbf{c}c_t^{(2)}\right]z$ and $\left[\mathbf{B}\mathbf{c}^{(2)} + \mathbf{d}c_t^{(2)}\right]z$, which are corresponding to the uniform heat flow in an infinite medium without holes, will not produce stress and strain, and thus can be cut out in the boundary equations. Keeping in mind that, (65) and (67) can be rewritten as

$$g(z) = c_t^{(1)}z + \gamma_1 \ln\zeta(z) + \gamma_2\zeta^{-2}(z) \tag{82}$$

$$\mathbf{f}(z) = \mathbf{c}^{(1)}z + \delta\ln\zeta(z) + \mathbf{f}_0(z) \tag{83}$$

Inserting (82) and (83) into (40) and (41) yields

$$\phi = 2\mathrm{Re}\left[\left(\mathbf{B}\mathbf{c}^{(1)} + \mathbf{d}c_t^{(1)}\right)z\right] + 2\mathrm{Re}[(\mathbf{B}\delta + \mathbf{d}\gamma_1)\ln\zeta] + 2\mathrm{Re}\left[\mathbf{B}\mathbf{f}_0(z) + \mathbf{d}\gamma_2\zeta^{-2}\right] \tag{84}$$

$$\mathbf{u} = 2\mathrm{Re}\left[\left(\mathbf{A}\mathbf{c}^{(1)} + \mathbf{c}c_t^{(1)}\right)z\right] + 2\mathrm{Re}[(\mathbf{A}\delta + \mathbf{c}\gamma_1)\ln\zeta] + 2\mathrm{Re}\left[\mathbf{A}\mathbf{f}_0(z) + \mathbf{c}\gamma_2\zeta^{-2}\right] \tag{85}$$

Obviously the first terms on the right of (84) and (85) stand for the complex potentials of an infinite medium without hole subjected to the uniform mechanical–electric loads at infinity. In this case, the electric-elastic fields in the medium are the same everywhere as those applied at infinity. This implies from (24) and (25) that the following identities hold

$$2\mathrm{Re}\left[\left(\mathbf{B}\mathbf{c}^{(1)} + \mathbf{d}c_t^{(1)}\right)z\right] = \phi^{\infty} \tag{86}$$

$$2\mathrm{Re}\left[\left(\mathbf{A}\mathbf{c}^{(1)} + \mathbf{c}c_t^{(1)}\right)z\right] = \mathbf{u}^{\infty} \tag{87}$$

where

$$\phi^{\infty} = \Pi_2^{\infty}x_1 - \Pi_1^{\infty}x_2 \tag{88}$$

$$\mathbf{u}^{\infty} = \varepsilon_1^{\infty}x_1 + \varepsilon_2^{\infty}x_2 \tag{89}$$

$$\Pi_1^{\infty} = \left(\sigma_{11}^{\infty}, \sigma_{12}^{\infty}, \sigma_{13}^{\infty}, D_1^{\infty}\right)^t = -\phi_{,2}^{\infty}$$

$$\Pi_2^{\infty} = \left(\sigma_{21}^{\infty}, \sigma_{22}^{\infty}, \sigma_{23}^{\infty}, D_2^{\infty}\right)^t = \phi_{,1}^{\infty}$$

$$\varepsilon_1^{\infty} = \mathbf{u}_{,1}^{\infty} = \left(\varepsilon_{11}^{\infty}, \varepsilon_{12}^{\infty} + \omega_3^{\infty}, 2\varepsilon_{13}^{\infty}, -E_1^{\infty}\right)^t$$

$$\varepsilon_2^{\infty} = \mathbf{u}_{,2}^{\infty} = \left(\varepsilon_{21}^{\infty} - \omega_3^{\infty}, \varepsilon_{22}^{\infty}, 2\varepsilon_{23}^{\infty}, -E_2^{\infty}\right)^t$$

Using (86) and (87), (84) and (85) become

$$\phi = \phi^{\infty} + 2\mathrm{Re}[(\mathbf{B}\delta + \mathbf{d}\gamma_1)\ln\zeta] + 2\mathrm{Re}\left[\mathbf{B}\mathbf{f}_0(z) + \mathbf{d}\gamma_2\zeta^{-2}\right] \tag{90}$$

$$\mathbf{u} = \mathbf{u}^{\infty} + 2\mathrm{Re}[(\mathbf{A}\delta + \mathbf{c}\gamma_1)\ln\zeta] + 2\mathrm{Re}\left[\mathbf{A}\mathbf{f}_0(z) + \mathbf{c}\gamma_2\zeta^{-2}\right] \tag{91}$$

On the hole, using $\zeta = \sigma = e^{i\theta}$ and (88), (89), (71) and (72) one has from (90) and (91) that

$$\phi(\sigma) = \Pi_2^{\infty}x_1(\sigma) - \Pi_1^{\infty}x_2(\sigma) + 2\mathrm{Re}\left[\mathbf{B}\hat{\mathbf{f}}_0(\sigma) + \mathbf{d}\gamma_2\sigma^{-2}\right] \tag{92}$$

$$\mathbf{u}(\sigma) = \varepsilon_1^{\infty}x_1(\sigma) + \varepsilon_2^{\infty}x_2(\sigma) + 2\mathrm{Re}\left[\mathbf{A}\hat{\mathbf{f}}_0(\sigma) + \mathbf{c}\gamma_2\sigma^{-2}\right] \tag{93}$$

Define a new function $\mathbf{K}_0(\zeta)$ as

$$\mathbf{K}_0(\zeta) = \mathbf{B}\hat{\mathbf{f}}_0(\zeta) + \mathbf{d}\gamma_2\zeta^{-2} \tag{94}$$

Then, (92) and (93) can be reduced into

$$\phi(\sigma) = \Pi_2^{\infty}x_1(\sigma) - \Pi_1^{\infty}x_2(\sigma) + 2\mathrm{Re}[\mathbf{K}_0(\sigma)] \tag{95}$$

$$\mathbf{u}(\sigma) = \varepsilon_1^{\infty}x_1(\sigma) + \varepsilon_2^{\infty}x_2(\sigma) + 2\mathrm{Im}\left[\mathbf{Y}\mathbf{K}_0(\sigma) - \mathbf{M}\gamma_2\sigma^{-2}\right] \tag{96}$$

where

$$\mathbf{Y} = i\mathbf{A}\mathbf{B}^{-1}, \quad \mathbf{M} = \mathbf{Y}\mathbf{d} - i\mathbf{c}$$

Once one obtains $\mathbf{K}_0(\zeta)$ from the given boundary condition, $\mathbf{f}_0(z)$ can be given by using (94), and then all the field variables can be determined without difficulty.

To find $\mathbf{K}_0(\zeta)$, one has to use the continuous conditions of the traction, the normal component of electric displacement and the electric potential on the hole. These conditions require from (42) and (43) that

$$\phi(\sigma) = \mathbf{i}_4 \int_s D_n^0 \, \mathrm{d}s \tag{97}$$

$$[\mathbf{u}(\sigma)]_4 = \varphi_h(\sigma) \tag{98}$$

where $[\ ]_4$ stands for taking the fourth row of the vector inside [ ], and $\mathbf{i}_4 = (0,0,0,1)^t$.

Substituting (95), (96), (43) and (47) into (97) and (98) yields

$$\Pi_2^\infty x_1(\sigma) - \Pi_1^\infty x_2(\sigma) + 2\mathrm{Re}[\mathbf{K}_0(\sigma)] = \mathbf{i}_4 2\varepsilon_0 \mathrm{Im}\hat{f}_h(\sigma) \tag{99}$$

$$\left[\varepsilon_1^\infty x_1(\sigma) + \varepsilon_2^\infty x_2(\sigma)\right]_4 + 2\mathrm{Im}\left[\mathbf{Y}\mathbf{K}_0(\sigma) - \mathbf{M}\gamma_2\sigma^{-2}\right]_4 = 2\mathrm{Re}\hat{f}_h(\sigma) \tag{100}$$

Substituting (59), (60) and (48) into (99) and (100), and then multiplying both sides of them by $\int_\gamma \mathrm{d}\sigma/(\sigma - \zeta)$, one can obtain after calculating the Cauchy integration that (Muskhelishvili, 1975)

$$\mathbf{K}_0(\zeta) = -\frac{1}{2}\left(a\Pi_2^\infty - ib\Pi_1^\infty\right)\zeta^{-1} - \mathbf{i}_4\varepsilon_0 i \sum_{n=1}^{\infty}\left(a_n^0 m_0^n - \overline{a_n^0}\right)\zeta^{-n} \tag{101}$$

$$-i\left[\mathbf{Y}\mathbf{K}_0(\zeta) - \mathbf{M}\gamma_2\zeta^{-2}\right]_4 = \frac{1}{2}\left(aE_1^\infty + ibE_2^\infty\right)\zeta^{-1} + \sum_{n=1}^{\infty}\left(a_n^0 m_0^n + \overline{a_n^0}\right)\zeta^{-n} \tag{102}$$

To find $a_n^0$ in (101), substituting (101) into (102), and then equating the coefficients of the same power $\zeta^{-n}$ in both sides of (102), one has

$$a_n^0 m_0^n (1 + \varepsilon_0 Y_{44}) + a_n^0 (1 - \varepsilon_0 Y_{44}) = c_n^0 \tag{103}$$

where

$$c_1^0 = \frac{1}{2}a\left[i\sum_{j=1}^{4} Y_{4j}\Pi_{2j}^\infty - E_1^\infty\right] + i\frac{1}{2}b\left[i\sum_{j=1}^{4} Y_{4j}\Pi_{1j}^\infty - E_2^\infty\right] \tag{104}$$

$$c_2^0 = i\gamma_2 M_4 = \frac{R_t m_t}{4\kappa_t}\left(aq_2^\infty - ibq_1^\infty\right)M_4 \tag{105}$$

$$c_n^0 = 0, \quad (n \geqslant 3) \tag{106}$$

Noting that $Y_{44}$ is real (Suo et al., 1992), (103) and its conjugal equation result in

$$a_n^0 = \frac{\overline{c_n^0} - m_0^n c_n^0 - \varepsilon_0 Y_{44}\left(\overline{c_n^0} + m_0^n c_n^0\right)}{\Delta_n}, \quad (n = 1, 2) \tag{107}$$

$$a_n^0 = 0, \quad (n \geqslant 3) \tag{108}$$

where

$$\Delta_n = \left(1 - m_0^{2n}\right)\left(1 + \varepsilon_0^2 Y_{44}^2\right) - 2\varepsilon_0 Y_{44}\left(1 + m_0^{2n}\right) \tag{109}$$

Substituting (107) and (108) into (101) and (48), one can finally obtain

$$\mathbf{K}_0(\zeta) = -\frac{1}{2}\left(a\Pi_2^\infty - ib\Pi_1^\infty\right)\zeta^{-1} - \mathbf{i}_4\varepsilon_0 i\Lambda_1\zeta^{-1} - \mathbf{i}_4\varepsilon_0 i\Lambda_2\zeta^{-2} \tag{110}$$

$$\hat{f}_h(\zeta) = \frac{a_1^0}{R_0}z + a_2^0\left(\zeta^2 + m_0^2\zeta^{-2}\right) \tag{111}$$

where

$$\Lambda_n = a_n^0 m_0^n - \overline{a_n^0}, \quad (n = 1, 2) \tag{112}$$

After $\mathbf{K}_0(\zeta)$ is determined, $\hat{\mathbf{f}}_0(\zeta)$ can be obtained from (94). With $\hat{\mathbf{f}}_0(\zeta)$ and $\hat{f}_h(\zeta)$, all the field variables can be calculated.

In addition, it can be seen that if letting the heat flow at infinity be zero, one has from (105) and (107) $a_2^0 = 0$, and (111) becomes

$$f_h(z) = \frac{a_1^0}{R_0} z \tag{113}$$

(113) implies that in this case, the electric field inside the elliptic hole is uniform. This is, however, not true in general cases.

If the hole is not very slender (e.g. $b/a > 10^{-1}$), the electric field inside the hole can be neglected, i.e. $\varepsilon_0$ can be assumed to be zero. In this case, (110) can be simplified to

$$\mathbf{K}_0(\zeta) = -\frac{1}{2}\left(a\Pi_2^\infty - ib\Pi_1^\infty\right)\zeta^{-1} \tag{114}$$

However, when the hole degenerates into a crack, the above simplification may lead to erroneous results, as it will be seen in the following analysis.

### 3.4. The general expressions of stresses on the hole rim

Let $\mathrm{d}s = \rho \mathrm{d}\theta$ be infinitesimal arc-length of the hole boundary $\Gamma$ where

$$\rho(\theta) = \left(a^2 \sin^2\theta + b^2 \cos^2\theta\right)^{1/2}$$

Then, the unit vectors tangential and normal to $\Gamma$, as shown in Fig. 1, can be expressed as

$$\mathbf{n}^t = \left(-\frac{\mathrm{d}x_1}{\mathrm{d}s}, -\frac{\mathrm{d}x_2}{\mathrm{d}s}, 0\right)$$

$$\mathbf{m}^t = \left(\frac{\mathrm{d}x_2}{\mathrm{d}s}, -\frac{\mathrm{d}x_1}{\mathrm{d}s}, 0\right) \tag{115}$$

where

$$x_1 = a\cos\theta, \ x_2 = b\sin\theta$$

On the other hand, we have

$$\mathbf{n}^t = (\cos\varphi, \sin\varphi, 0)$$

$$\mathbf{m}^t = (-\sin\varphi, \cos\varphi, 0) \tag{116}$$

Comparing (115) with (116), one can obtain the relation between $\varphi$ and $\theta$ as

$$\cos\varphi = \frac{a\sin\theta}{\rho(\theta)}$$

$$\sin\varphi = -\frac{b\cos\theta}{\rho(\theta)} \tag{117}$$

Letting $\mathbf{t}_m$ and $\mathbf{t}_n$ be the generalized traction on the hole surface and on the surface perpendicular to $\Gamma$, respectively, see Fig. 1, one has

$$\mathbf{t}_m = -\phi_{,n} = -(\phi_{,1}\cos\varphi + \phi_{,2}\sin\varphi) \tag{118}$$

$$\mathbf{t}_n = -\phi_{,m} = \phi_{,1}\sin\varphi - \phi_{,2}\cos\varphi \tag{119}$$

Then, the hoop stress $\sigma_{nn}$ and the two shear stresses $\tau_{nm}$ and $\tau_{n3}$ can be written as

$$\sigma_{nn} = \mathbf{n}_*^t \cdot \mathbf{t}_n, \quad \tau_{nm} = \mathbf{m}_*^t \cdot \mathbf{t}_n, \quad \tau_{n3} = \mathbf{i}_3^t \cdot \mathbf{t}_n \tag{120}$$

where

$$\mathbf{n}_*^t = (\mathbf{n}^t, 0), \quad \mathbf{m}_*^t = (\mathbf{m}^t, 0), \quad \mathbf{i}_3^t = (0, 0, 1, 0)$$

Similarly, the hoop component of electric displacement is

$$D_n = \mathbf{i}_4^t \cdot \mathbf{t}_n \tag{121}$$

Obviously, if $\phi$ is obtained, the stress and electric displacement along the hole boundary can be calculated from Eqs. (120) and (121).

### 3.5. Numerical examples

Consider a transversly isotropic piezoelectric medium with an elliptic cavity and introduce a material coordinate system $(X_1, X_2, X_3)$, where the poling direction is parallel to the $X_3$-axis. If our attention is focused on the field in $X_1 - X_3$ plane, the out-of plane displacement does not couple with the in-plane displacements and the electric potential, and the elastic matrices **S**, **R** and **W** degenerate into the $3 \times 3$ ones:

$$\mathbf{S} = \begin{bmatrix} c_{11} & 0 & 0 \\ 0 & c_{44} & e_{15} \\ 0 & e_{15} & -\varepsilon_{11} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 0 & c_{13} & e_{31} \\ c_{44} & 0 & 0 \\ e_{15} & 0 & 0 \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} c_{44} & 0 & 0 \\ 0 & c_{33} & e_{33} \\ 0 & e_{33} & -\varepsilon_{33} \end{bmatrix}$$

and

$$\beta_1 = (\beta_{11}, 0, 0)^t, \quad \beta_2 = (0, \beta_{33}, \tau_3)^t$$

Assuming the considered medium to be cadmium selenide, the elements of **S**, **R** and **W** can be determined from the following material constants (Ashida et al., 1997):

Elastic constants:

$$c_{11} = 74.1 \times 10^9\ \mathrm{Nm^{-2}}, \quad c_{13} = 39.3 \times 10^9\ \mathrm{Nm^{-2}}, \quad c_{12} = 45.2 \times 10^9\ \mathrm{Nm^{-2}}$$
$$c_{33} = 83.6 \times 10^9\ \mathrm{Nm^{-2}}, \quad c_{44} = 13.2 \times 10^9\ \mathrm{Nm^{-2}}$$

Piezoelectric constants:

$$e_{31} = -0.16\ \mathrm{cm^{-2}}, \quad e_{33} = 0.347\ \mathrm{cm^{-2}}, \quad e_{15} = -0.138\ \mathrm{cm^{-2}}$$

Dielectric constants:

$$\varepsilon_{11} = 82.6 \times 10^{-12}\ \mathrm{C^2/Nm^2}, \quad \varepsilon_{33} = 90.3 \times 10^{-12}\ \mathrm{C^2/Nm^2}, \quad \varepsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C^2/Nm^2}$$

Pyroelectric constant: $\tau_3 = -2.94 \times 10^{-6}\ \mathrm{C/Km^2}$

Stress-temperature constants:

$$\beta_{11} = 0.621 \times 10^6\ \mathrm{N/Km^2}, \quad \beta_{33} = 0.551 \times 10^6\ \mathrm{N/Km^2}$$

Since the coefficients of thermal conductivity for cadmium selenide could not be found in the literature, the following values are assumed: $\lambda_{13} = 0$, $\lambda_{11} = \lambda_{22} = 1 \times 10^{-6}\ \mathrm{m^2/s}$ and $\lambda_{33} = 1.5\lambda_{11}$.

Note that the above material constants are represented in the material coordinate system $(X_1, X_2, X_3)$. For convenience, the $X_1 - X_3$ plane is relabelled as the $x_1 - x_2$ plane in the following analyses. Consequently, the numerical calculation of stress and electric displacement on the hole rim can be conducted by using the formulations in the above Sections. Firstly, based on the given constants, we have

$$\mu_1 = 1.8267\mathrm{i}, \quad \mu_2 = 0.8303\mathrm{i}, \quad \mu_3 = 0.5941\mathrm{i}, \quad \mu_t = 0.816\mathrm{i}$$

Then, the hoop components of stress and electric displacement along the rim of the hole are plotted, respectively, for the different loading cases, as shown in Figs. 2–8.

Figs. 2 and 3 show that when the mechanical load is solely applied at infinity, the influence of $\varepsilon_0$ is very small on the stress and electric displacement on the rim of hole. Especially when $a/b < 10$, the electric field within the hole can be neglected, that is, the impermeable boundary condition can be adopted to simplify the analysis. However, when the remote electric load is applied, as it is found from Figs. 4–6, $\varepsilon_0$ has great influence on the electric displacement. For example, as $a/b$ increases, as shown in Fig. 5, the results based on the exact boundary condition show the maximum hoop component of electric displacement $D_{\theta\max}$ approaches a constant ($D_{\theta\max} = D_2^\infty$) at $\theta = 0$ or $\pi$, while $D_{\theta\max}$ exhibits the nature of singularity if the assumption $\varepsilon_0 = 0$ is used.

When the heat flux is exerted at infinity, it is shown from Figs. 7 and 8 that both the concentration factors of stress and electric displacement increase as $a/b$ increases, no matter if $\varepsilon_0$ is or not assumed to be zero. These numerical results show that when no electric load exists, the impermeable boundary condition is valid to the case of non-slender hole (e.g. $a/b < 10$ ). However, when there is an electric loading, this condition may lead to erroneous results, specially when the home becomes a crack.

## 4. The solution to a crack

### 4.1. The complex potential

When the elliptic hole degenerates into a crack along the $x_1$-axis, letting $m_0 = m_t = m_\alpha = 1$ and $R_0 = R_t = R_\alpha = a/2$, one has from (109 ) and (112) that

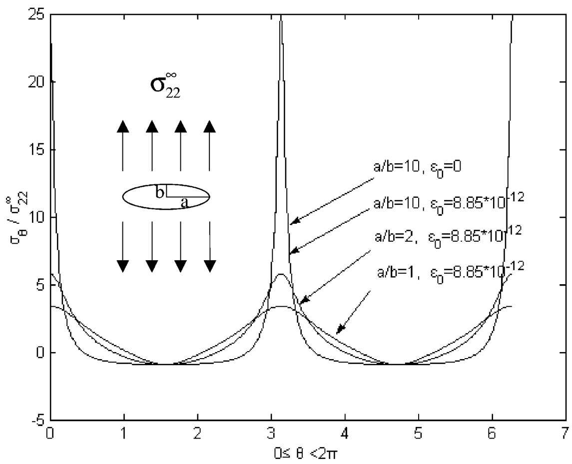


Fig. 2. The variation of the hoop stress with $\theta$ when $\sigma_{22}^\infty$ is solely applied.

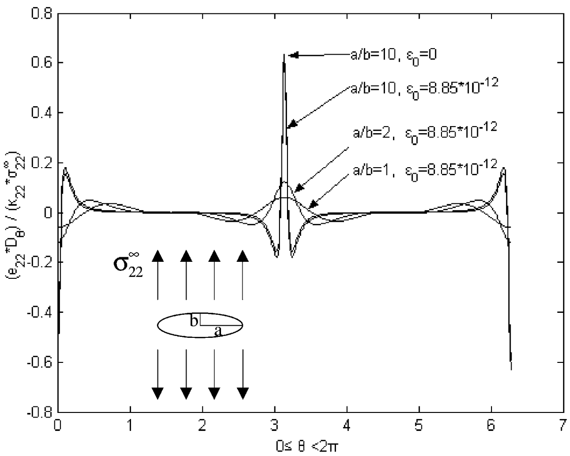


Fig. 3. The variation of the hoop electric displacement with $\theta$ when $\sigma_{22}^{\infty}$ is solely applied.

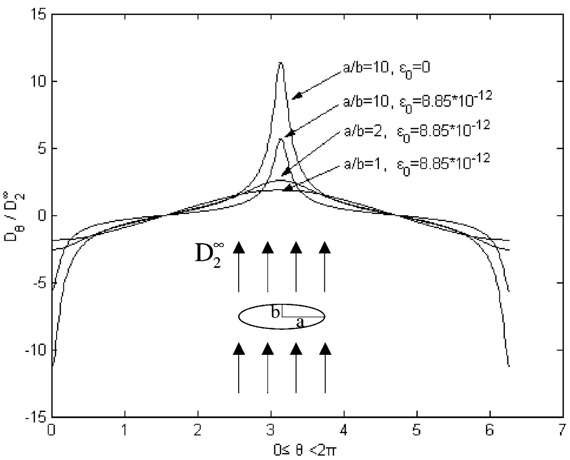


Fig. 4. The variation of the hoop electric displacement with $\theta$ when $D_2^{\infty}$ is solely applied.

$$\Delta_n = -4\varepsilon_0 Y_{44}, \quad \Lambda_n = 2\mathrm{i}\mathrm{Im}\left[a_n^0\right], \quad (n = 1, 2) \tag{122}$$

Using (107), one obtains

$$\mathrm{Im}\left[a_n^0\right] = -\frac{2}{\Delta_n}\mathrm{Im}\left[c_n^0\right] \tag{123}$$

Similarly, one has

$$\mathrm{Re}\left[a_n^0\right] = \tfrac{1}{2}\mathrm{Re}\left[c_n^0\right] \tag{124}$$

Substituting (123) with $(122)_1$ into $(122)_2$ produces

$$\Lambda_n = \frac{i}{\varepsilon_0 Y_{44}}\mathrm{Im}\left[c_n^0\right] \tag{125}$$

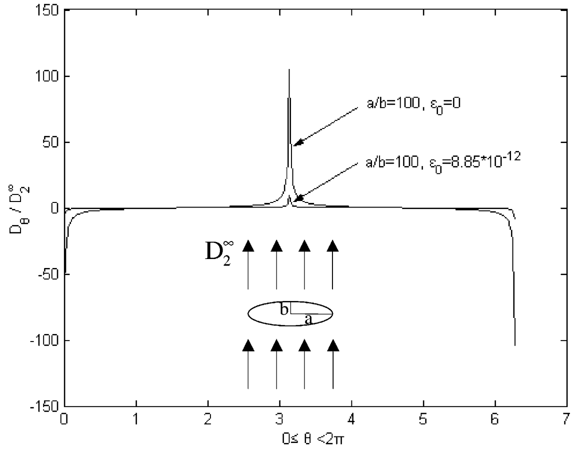


Fig. 5. The variation of the hoop electric displacement with $\theta$ when $D_2^\infty$ is solely applied ($a/b = 100$).

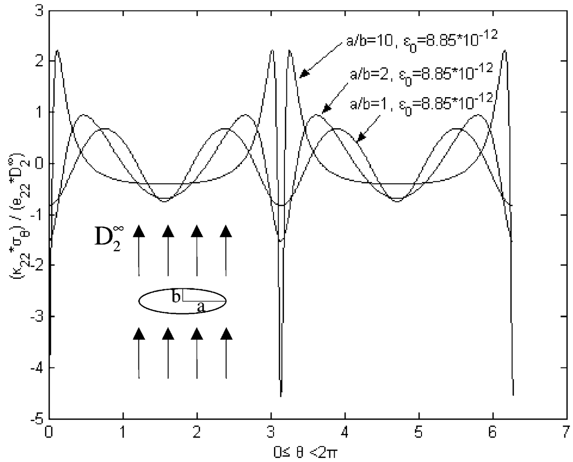


Fig. 6. The variation of the hoop stress with $\theta$ when $D_2^\infty$ is solely applied.

Inserting (104) and (105) into (125), one finds

$$\Lambda_1 = \frac{ia}{2\varepsilon_0 Y_{44}}\left(\sum_{j=1}^{4} Y_{4j}\Pi_{2j}^\infty\right), \quad \Lambda_2 = \frac{ia^2 \mathrm{Im} M_4}{4\varepsilon_0 \kappa_t Y_{44}} q_2^\infty \tag{126}$$

Substituting (126) into (110), one can obtain the final expression of $\mathbf{K}_0(\zeta)$ for the case of a crack. The result is

$$\mathbf{K}_0(\zeta) = -\frac{1}{2} a\Pi_2^\infty \zeta^{-1} + \mathbf{i}_4 \frac{a}{2Y_{44}}\left(\sum_{j=1}^{4} Y_{4j}\Pi_{2j}^\infty\right)\zeta^{-1} + \mathbf{i}_4 \frac{a^2 \mathrm{Im} M_4}{4\kappa_t Y_{44}} q_2^\infty \zeta^{-2} \tag{127}$$

With $\mathbf{K}_0(\zeta)$, the complex potential $\mathbf{f}_0(z)$ can be found from (94).

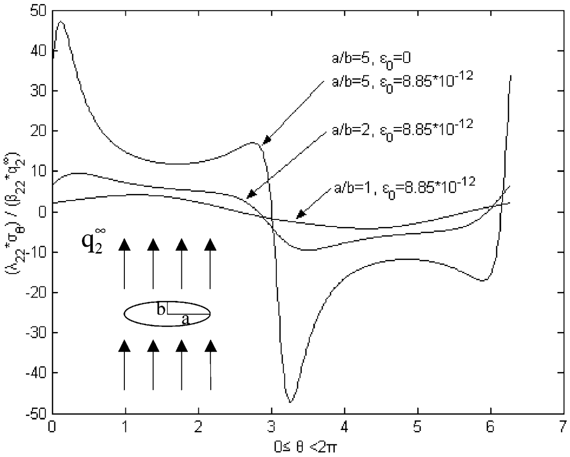


Fig. 7. The variation of the hoop stress with $\theta$ when $q_2^\infty$ is solely applied.

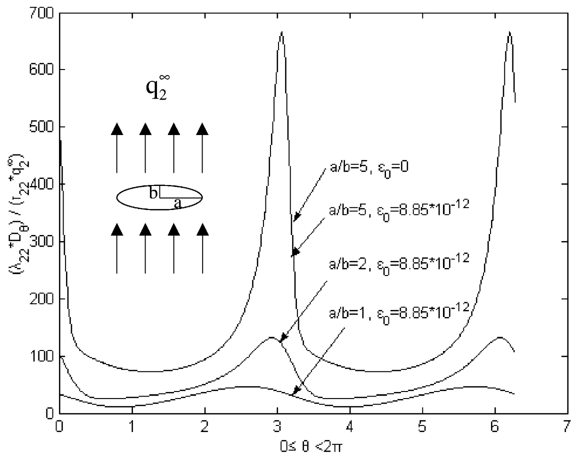


Fig. 8. The variation of the hoop electric displacement with $\theta$ when $q_2^\infty$ is solely applied.

## 4.2. The field intensity factor

Define

$$\mathbf{K} = (k_{\mathrm{II}}, k_{\mathrm{I}}, k_{\mathrm{III}}, k_D)^t$$

Then, at the right crack tip ($x_1 = a$), the field intensity factor vector $\mathbf{K}$ can be expressed as

$$\mathbf{K}(a) = \sqrt{2\pi}\lim_{z\to a}(z-a)^{\frac{1}{2}}\phi_{,1}(z) \tag{128}$$

In $\zeta$-plane, by using (57), (128) can be rewritten as

$$\mathbf{K}(a) = \sqrt{\frac{\pi}{a}} \lim_{\zeta \to 1} \phi(\zeta)/\mathrm{d}\zeta \tag{129}$$

Substituting (90) into (129), and noting from (72) that $\mathbf{B}\delta + \mathbf{d}\gamma_1$ is real, one has

$$\mathbf{K}(a) = 2\sqrt{\frac{\pi}{a}}[\mathbf{B}\delta + \mathbf{d}\gamma_1] + 2\sqrt{\frac{\pi}{a}} \mathrm{Re} \lim_{\zeta \to 1} \mathbf{K}_0(\zeta)/\mathrm{d}\zeta \tag{130}$$

Inserting (120) into (123), one can obtain the final solution of field intensity factor vector as

$$\mathbf{K}(a) = \mathbf{K}_T(a) + \mathbf{K}_\sigma(a) + \mathbf{K}_D^\sigma(a) + \mathbf{K}_D^T(a) \tag{131}$$

where

$$\mathbf{K}_T(a) = 2\sqrt{\frac{\pi}{a}}[\mathbf{B}\delta + \mathbf{d}\gamma_1] \tag{132}$$

$$\mathbf{K}_\sigma(a) = \sqrt{\pi a}\Pi_2^\infty \tag{133}$$

$$\mathbf{K}_D^\sigma(a) = -\mathbf{i}_4 \sqrt{\pi a} \frac{1}{Y_{44}} \sum_{j=1}^{4} \mathrm{Re}\left[Y_{4j}\right] \Pi_{2j}^\infty \tag{134}$$

$$\mathbf{K}_D^T(a) = -\mathbf{i}_4 \sqrt{\pi a} \frac{\mathrm{Im} M_4}{\kappa_t Y_{44}} a q_2^\infty \tag{135}$$

Observing (131), one can find that $\mathbf{K}(a)$ consists of the four parts, in which $\mathbf{K}_T(a)$ and $\mathbf{K}_\sigma(a)$ stand for the contribution of the applied heat loads and mechanical-electric loads to all the field singularities, respectively; while $\mathbf{K}_D^\sigma(a)$ results from the coupling effect between mechanical and electrical fields, and $\mathbf{K}_D^T(a)$ the coupling effect between thermal and electrical fields. Thus, $\mathbf{K}_D^\sigma(a)$ and $\mathbf{K}_D^T(a)$ represent the contribution of the applied mechanical–electrical and heat loads to the singularity of electric displacement field, respectively.

However, if the crack is assumed to be impermeable, i.e., $\varepsilon_0$ is assumed to be zero, one can find from (110) that the above coupling terms $\mathbf{K}_D^\sigma(a)$ and $\mathbf{K}_D^T(a)$ are omitted. Obviously this will lead to erroneous results.

Below let us examine several special cases. When there is no heat load at infinity, one can obtain from (131)–(135) that

$$K_j(a) = \sqrt{\pi a}\sigma_{2j}^\infty, \quad (j = 1, 2, 3); \qquad K_4(a) = -\sqrt{\pi a} \frac{1}{Y_{44}} \sum_{j=1}^{3} \mathrm{Re}\left[Y_{4j}\right] \sigma_{2j}^\infty \tag{136}$$

which is consistent with the result of Gao and Wang (2000).When the electric loads are solely applied at infinity, (129) leads to

$$\mathbf{K}(a) = \mathbf{0} \tag{137}$$

When the heat flow is solely applied at infinity, (131) becomes

$$\mathbf{K}(a) = \mathbf{K}_T(a) + \mathbf{K}_D^T(a) \tag{138}$$

(138) shows that in this case, both the stress and electric field are singular.

For a purely-elastic anisotropic material subjected only to the remote heat flow, neglecting the terms related to mechanical–electrical loads and noting $M_4 = 0$, one has from (131) that

$$\mathbf{K}(a) = 2\sqrt{\frac{\pi}{a}}[\mathbf{B}\delta + \mathbf{d}\gamma_1] \tag{139}$$

where **B** degenerates into a $3 \times 3$ matrix, **d** a $3 \times 1$ vector. It can be confirmed that (139) is consistent with the solution of Tarn and Wang (1993). Recently, this solution is also obtained by Chao and Shen (1998) who studied thermal stresses in a generally anisotropic body with an elliptic inclusion. However, it should be noted that there are some typing errors in Eqs. (64) and (65) of Chao and Shen (1998), i.e. the positive sign in these equations should be changed into the negative sign.

### *4.3. The electric field inside the crack*

From (111), one has

$$f_h(z) = \frac{a_1^0}{R_0}z + \frac{a_2^0}{R_0^2}z^2 - 2a_2^0 \tag{140}$$

Substituting (140) into $(45)_2$ and $(44)_1$ gives

$$D_2^0(x_1) = \frac{4\varepsilon_0}{a}\mathrm{Im}\left[a_1^0\right] + \frac{16\varepsilon_0}{a^2}\mathrm{Im}\left[a_2^0\right]x_1 \tag{141}$$

$$E_1^0(x_1) = -\frac{4}{a}\mathrm{Re}\left[a_1^0\right] - \frac{16}{a^2}\mathrm{Re}\left[a_2^0\right]x_1 \tag{142}$$

Inserting (123) and (124) into (141) and (142) leads to

$$D_2^0(x_1) = D_2^\infty + \frac{1}{Y_{44}}\sum_{j=1}^{3}\mathrm{Re}\left[Y_{4j}\right]\sigma_{2j}^\infty + \frac{\mathrm{Im}M_4}{\kappa_t Y_{44}}q_2^\infty x_1 \tag{143}$$

$$E_1^0(x_1) = E_1^\infty + \sum_{j=1}^{3}\mathrm{Im}\left[Y_{4j}\right]\sigma_{2j}^\infty - \frac{\mathrm{Re}M_4}{\kappa_t}q_2^\infty x_1 \tag{144}$$

If the mechanical loads are solely applied at infinity, (143) and (144) becomes

$$D_2^0 = D_2^\infty + \frac{1}{Y_{44}}\sum_{j=1}^{3}\mathrm{Re}\left[Y_{4j}\right]\sigma_{2j}^\infty \tag{145}$$

$$E_1^0 = E_1^\infty + \sum_{j=1}^{3}\mathrm{Im}\left[Y_{4j}\right]\sigma_{2j}^\infty \tag{146}$$

Comparing the above results concerning crack with those in Gao and Wang (2001), it can be found that they are consistent, though two different approaches are used. This implies that the Parton assumption is also valid to the mathematical crack problem in thermopiezoelectric solid. Thus, the Parton assumption can be directly used to solve a number of complicated crack problems in thermopiezoelectric materials.

## 5. Conclusions

This paper presents exact solutions for an infinite thermopiezoelectric medium with an insulated elliptic hole or a crack under combined heat–mechanical–electrical loadings at infinity. Even though there are the mathematical complexities inherent to the considered problem, the present analysis is very explicit. Especially when the hole degenerates into a crack, more concise results are obtained. Since these results are not only concise, but also exact, they can be used as the fundamental solutions to prove the correctness of other solutions for more complicated crack problems of thermopiezoelectric media. For example, this work shows that for the crack problem in thermopiezoelectric solids, the solutions based on the Parton assumption is right. This indicates that one can exactly and effectively solve the linear crack problem in piezoelectric media by the direct use of the Parton assumption.

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## References

Ashida, F., Choi, J.S., Noda, N., 1997. Control of elastic displacement in piezoelectric-based intelligent plate subjected to thermal load. Int. J. Eng. Sci. 35, 851–868.

Chandrasekharaiah, D.S., 1988. A generalized linear thermoelasticity theory for piezoelectric media. Acta Mech. 71, 39–49.

Chao, C.K., Shen, M.H., 1993. Explicit solutions for curvilinear cracks in the thermoelastic medium. J. Therm. Stresses 16, 215–231.

Chao, C.K., Chang, R.C., 1994. Thermoelastic problem of dissimilar anisotropic solids with a rigid line inclusion. ASME J. Appl. Mech. 61, 978–980.

Chao, C.K., Shen, M.H., 1997. On bonded circular inclusions in plane thermoelasticity. ASME J. Appl. Mech. 64, 1000–1004.

Chao, C.K., Shen, M.H., 1998. Thermal stresses in a generally anisotropic body with an elliptic inclusion subject to uniform heat flow. ASME J. Appl. Mech. 65, 51–58.

Chung, M.Y., Ting, T.C.T., 1996. Piezoelectric solid with an elliptic inclusion or hole. Int. J. Solid Struct. 33, 2343–3361.

Cutis, J.H., 1971. Faber polynomials and the Faber series. Am. Math. Mon. 78, 577–596.

Dunn, M.L., 1994. The effect of crack face boundary conditions on the fracture mechanics of piezoelectric solids. Eng. Fract. Mech. 48, 25–39.

Florence, A.L., Goodier, J.N., 1960. Thermal stresses due to disturbance of uniform heat flow by an insulated ovaloid hole. ASME J. Appl. Mech. 27, 635–639.

Gao, C.F., Fan, W.X., 1998. Green's functions for generalized 2D problems in piezoelectric media with an elliptical hole or a crack. Mech. Res. Commun. 25, 685–693.

Gao, C.F., Fan, W.X., 1999. Exact solutions for the plane problem in piezoelectric materials with an elliptic hole or a crack. Int. J. Solid Struct. 36, 2527–2540.

Gao, C.F., Wang, M.Z., 2000. Collinear permeable cracks between dissimilar piezoelectric materials. Int. J. Solid Struct. 37, 4969–4986.

Gao, C.F., 2000. Further study on the generalized 2D problem of an elliptical hole or a crack in piezoelectric media. Mech. Res. Commun. 27 (4), 429–434.

Gao, C.F., Wang, M.Z., 2001. Collinear permeable cracks in thermopiezoelectric materials. Mech. Mater. 33, 1–9.

Gao, C.F., 2001. Comments on General solutions for thermopiezoelectrics with various holes under thermal loading. Int. J. Solid Struct. 38, 6269–6271 (2000. Int. J. Solid Struct. 37, 5561–5578).

Hwu, C., 1990. Thermal stresses in an anisotropic plate disturbed by an insulated elliptic hole or crack. ASME J. Appl. Mech. 57, 916–922.

Hwu, C., 1992. Thermoelastic interface crack problems in dissimilar anisotropic media. Int. J. Solid Struct. 29, 2077–2090.

Kaminskii, A.A., Flegantov, L.A., 1994. Temperature stresses around cracks in an elastic plate weakened by two holes. Int. Appl. Mech. 30, 707–712.

Kattis, M.A., Patia, A.P., 1994. Thermal stress problem for a partly debonded rigid circular-arc fiber inclusion in an infinite matrix. Eng. Fract. Mech. 48, 359–366.

Kogan, L.C., Hui, Y., Molkov, V., 1996. Stress and induction field of a spheroidal inclusion or a penny-shaped crack in a transversely isotropic piezoelectric material. Int. J. Solid Struct. 33, 2719–2737.

Kosmodamianskii, A.S., Chemie, V.I., 1981. Stress state of a plate weakened by two elliptical holes with parallel axes. Soviet Appl. Mech. 17, 570–581.
Lee, K.Y., Jang, Y.K., 1993. Thermal stress analysis of an infinite plane containing a rigid inclusion of arbitrary shape. Eng. Fract. Mech. 46, 709–714.
Lin, S.T., Feng, Z., Rowlands, R.E., 1997. Thermoelastic determination of stress intensity factors in orthotropic composites using the J-integral. Eng. Fract. Mech. 56, 579–592.
McMeeking, R.M., 1989. Electrostrictive stresses near crack-like flaws. J. Appl. Math. Phys. (ZAMM) 640, 615–627.
Mindlin, R.D., 1974. Equations of high frequency vibrations of thermooizoelectric crystal plates. Int. J. Solid Struct. 10, 625–637.
Muskhelishvili, N.I., 1975. Some basic problems of mathematical theory of elasticity. Noordhoof, Leyden.
Pak, Y.E., Tobin, A., 1993. In: On electric field effects in fracture of piezoelectric materials, AMD-Vol. 161/MD-Vol. 42. Mechanics of Electromagnetic Materials and Structure, ASME, pp. 51–62.
Parton, V.Z., 1976. Fracture mechanics of piezoelectric materials. Acta Astronaut. 3, 671–683.
Qin, Q.H., Mai, Y.W., 1997. Crack growth prediction of an inclined crack in a half-plane thermopiezoelectric solid. Theor. Appl. Fract. Mech. 26, 185–191.
Qin, Q.H., Mai, Y.W., Yu, S.W., 1999. Some problems in plane thermopiezoelectric materials with holes. Int. J. Solid Struct. 36, 427–439.
Shang, F., Wang, Z., Li, Z., 1996. Thermal stresses analysis of a three-dimensional crack in a thermopiezoelectric solid. Eng. Fract. Mech. 55, 737–750.
Shen, S., Kuang, Z.B., 1998. Interface crack in bi-piezothermoelastic media and the interaction with a point heat source. Int. J. Solid Struct. 35, 3899–3915.
Sih, G.C., 1962. On the singular character of thermal stresses near a crack tip. ASME J. Appl. Mech. 29, 587–589.
Sosa, H.A., Khutoryansky, N., 1996. New developments concerning piezoelectric materials with defects. Int. J. Solid Struct. 33, 3399–3414.
Sturla, F.A., Barber, J.R., 1988. Thermal stresses due to a plane crack in general anisotropic material. ASME J. Appl. Mech. 55, 372–376.
Suo, Z., 1990. Singularities interfaces and cracks in dissimilar anisotropic media. Proc. R. Soc. Lond. A 427, 331–358.
Suo, Z., Kuo, C.M., Barnett, D.M., Willis, J.R., 1992. Fracture mechanics for piezoelectric ceramics. J. Mech. Phys. Sol. 40, 739–765.
Tarn, J.Q., Wang, Y.M., 1993. Thermal stresses in anisotropic bodies with a hole or a rigid inclusion. J. Therm. Stresses 16, 455–471.
Ting, T.C.T., 2000. Common errors on mapping of nonelliptic curves in anisotropic elasticity. ASME J. Appl. Mech. 67, 655–657.
Wu, C.H., 1984. Plane anisotropic thermoelasticity. ASME J. Appl. Mech. 51, 724–726.
Yang, X.X., Shen, S., Kuang, Z.B., 1997. The degenerate solution for piezothermoelastic materials. Eur. J. Mech. A/Solid 16, 779–793.
Yu, S.W., Qin, Q.H., 1996. Damage analysis of thermopiezoelectric properties: Part I-crack tip singularities. Theor. Appl. Fract. Mech. 25, 263–277.
Zhang, X.Z., Hasebe, N., 1993. Basic singular thermoelastic solutions for a crack. Int. J. Fract. 62, 97–118.
Zhang, T.Y., Qian, C.F., Tong, P., 1998. Linear electro-elastic analysis of a cavity or a crack in a piezoelectric material. Int. J. Solid Struct. 35, 2121–2149.